On Chapter 7 of *Laws of Form*

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1 Introduction

In chapter 7 of *Laws of Form* Spencer-Brown extends the scope of his basic equations to expressions with any finite number of variables. Some of his arguments, when he provides them, are rigorous; others are mere sketches, and some possible generalizations are left unmentioned. This paper will present fully rigorous proofs of the propositions.

Below is a list of axioms and theorems referenced in subsequent proofs:

$$\overline{\overline{pr}\rceil\ \overline{qr}\rceil}\rceil = \overline{\overline{p}\rceil\ \overline{q}\rceil}\rceil r \tag{J2}$$

$$\overline{pr}\rceil\ \overline{qr}\rceil = \overline{\overline{\overline{p}\rceil\ \overline{q}\rceil}\rceil r}\rceil \tag{J2.1}$$

$$\overline{\overline{a}\rceil}\rceil = a \tag{C1}$$

$$\overline{ab}\rceil b = \overline{a}\rceil b \tag{C2}$$

$$\overline{\overline{\overline{a}\rceil b}\rceil\ c}\rceil = \overline{ac}\rceil\ \overline{\overline{b}\rceil c}\rceil \tag{C7}$$

$$\overline{\overline{a}\rceil\ \overline{br}\rceil\ \overline{cr}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{b}\rceil\ \overline{c}\rceil}\rceil\ \overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil \tag{C8}$$

$$\overline{\overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{b}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{x}\rceil r}\rceil\ \overline{\overline{y}\rceil r}\rceil}\rceil = \overline{r}\rceil ab\rceil\ \overline{rxy}\rceil \tag{C9}$$

$$\overline{\overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{x}\rceil\ \overline{r}\rceil}\rceil}\rceil = \overline{r}\rceil\ \overline{a}\rceil\ \overline{rx}\rceil \tag{C9.1}$$

2 General theorems

Spencer-Brown begins the chapter by sketching an inductive generalization of J2. Here is the proof in full.

Theorem (J2*).

$$\overline{\overline{a_1}\rceil\ \overline{a_2}\rceil\ \ldots\ \overline{a_n}\rceil}\rceil r = \overline{\overline{a_1 r}\rceil\ \overline{a_2 r}\rceil\ \ldots\ \overline{a_n r}\rceil}\rceil$$

Proof. The proof proceeds by induction on n. The base case is J2, where $n = 2$. Let the induction hypothesis (J2h) be:

$$\overline{\overline{a_1}\rceil\ \overline{a_2}\rceil\ \ldots\ \overline{a_n}\rceil}\rceil r = \overline{\overline{a_1 r}\rceil\ \overline{a_2 r}\rceil\ \ldots\ \overline{a_n r}\rceil}\rceil$$

The induction step:

$$
\begin{aligned}
&\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|\ \overline{a_{n+1}}|}|\ r \\
&= \overline{\overline{\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|}|}|\ \overline{a_{n+1}}|}|\ r && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|}|\ r}|\ \overline{a_{n+1} r}|}| && \text{(J2)} \\
&= \overline{\overline{\overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}|}|}|\ \overline{a_{n+1} r}|}| && \text{(J2h)} \\
&= \overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}|\ \overline{a_{n+1} r}|}| && \text{(C1)}
\end{aligned}
$$

□

Alternate proof. A very similar and equally short proof, using the same induction hypothesis as above. The induction step:

$$
\begin{aligned}
&\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|\ \overline{a_{n+1}}|}|\ r \\
&= \overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{\overline{\overline{a_n}|\ \overline{a_{n+1}}|}|}|}|\ r && \text{(C1)} \\
&= \overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{\overline{\overline{a_n}|\ \overline{a_{n+1}}|}|\ r}|}| && \text{(J2h)} \\
&= \overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{\overline{\overline{a_n r}|\ \overline{a_{n+1} r}|}|}|}| && \text{(J2)} \\
&= \overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}|\ \overline{a_{n+1} r}|}| && \text{(C1)}
\end{aligned}
$$

□

Before continuing, I prove a useful generalization of corollary J2.1.

Theorem (J2.1*).

$$\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}| = \overline{\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|}|\ r}|$$

Proof.

$$
\begin{aligned}
&\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}| \\
&= \overline{\overline{\overline{a_1 r}|\ \overline{a_2 r}|\ \ldots\ \overline{a_n r}|}|}| && \text{(C1)} \\
&= \overline{\overline{\overline{a_1}|\ \overline{a_2}|\ \ldots\ \overline{a_n}|}|\ r}| && \text{(J2*)}
\end{aligned}
$$

□

Spencer-Brown states the generalizations of C8 and C9 but omits the proofs, merely noting that they are similar to J2*.

Theorem (C8*).

$$\overline{\overline{a}|\ \overline{b_1 r}|\ \overline{b_2 r}|\ \ldots\ \overline{b_n r}|}| = \overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \ldots\ \overline{b_n}|}|\ \overline{\overline{a}|\ \overline{r}|}|$$

Proof. The proof proceeds by induction on n. The base case is C8, where $n = 2$. Let the induction hypothesis (C8h) be:

$$\overline{\overline{a}|\ \overline{b_1r}|\ \overline{b_2r}|\ \dots\ \overline{b_nr}|}| = \overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|}|\ \overline{\overline{a}|\ \overline{r}|}|$$

The induction step:

$$\begin{aligned}
&\overline{\overline{a}|\ \overline{b_1r}|\ \overline{b_2r}|\ \dots\ \overline{b_nr}|\ \overline{b_{n+1}r}|}| \\
&= \overline{\overline{\overline{\overline{a}|\ \overline{b_1r}|}|}|\ \overline{b_2r}|\ \dots\ \overline{b_nr}|\ \overline{b_{n+1}r}|}| && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a}|\ \overline{b_1r}|}|}|\ \overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ \overline{\overline{\overline{\overline{a}|\ \overline{b_1r}|}|}|\ \overline{r}|}| && \text{(C8h)} \\
&= \overline{\overline{\overline{\overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ \overline{\overline{r}|}|}|\ \overline{\overline{\overline{a}|\ \overline{b_1r}|}|}|}| && \text{(J2.1)} \\
&= \overline{\overline{\overline{\overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ r}|\ \overline{a}|\ \overline{b_1r}|}| && \text{(C1 twice)} \\
&= \overline{\overline{\overline{\overline{\overline{\overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|}|\ \overline{b_1}|}|\ r}|\ \overline{a}|}| && \text{(J2.1)} \\
&= \overline{\overline{\overline{\overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ r}|\ \overline{a}|}| && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ \overline{\overline{r}|}|}|\ \overline{a}|}| && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ \overline{\overline{a}|\ \overline{r}|}|}|}| && \text{(J2)} \\
&= \overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|\ \overline{b_{n+1}}|}|\ \overline{\overline{a}|\ \overline{r}|}| && \text{(C1)}
\end{aligned}$$

□

J2.1* allows for a quicker direct proof.

Alternate proof.

$$\begin{aligned}
&\overline{\overline{a}|\ \overline{b_1r}|\ \overline{b_2r}|\ \dots\ \overline{b_nr}|}| \\
&= \overline{\overline{a}|\ \overline{\overline{\overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|}|\ r}|}| && \text{(J2.1*)} \\
&= \overline{\overline{a}|\ \overline{\overline{\overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|}|\ \overline{\overline{r}|}|}|}| && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|}|\ \overline{\overline{a}|\ \overline{r}|}|}|}| && \text{(J2)} \\
&= \overline{\overline{a}|\ \overline{b_1}|\ \overline{b_2}|\ \dots\ \overline{b_n}|}|\ \overline{\overline{a}|\ \overline{r}|}| && \text{(C1)}
\end{aligned}$$

□

Theorem (C9*).

$$\overline{\overline{\overline{\overline{a_1}|\;\overline{r}|}|\;\overline{\overline{a_2}|\;\overline{r}|}|\ldots\overline{\overline{a_n}|\;\overline{r}|}|\;\;\overline{\overline{x_1}|\,r}|\;\overline{\overline{x_2}|\,r}|\ldots\overline{\overline{x_m}|\,r}|}|}|$$
$$= \overline{\overline{r}|\,a_1a_2\ldots a_n}|\;\overline{rx_1x_2\ldots x_m}|$$

Proof.

$$\overline{\overline{\overline{\overline{a_1}|\;\overline{r}|}|\;\overline{\overline{a_2}|\;\overline{r}|}|\ldots\overline{\overline{a_n}|\;\overline{r}|}|\;\;\overline{\overline{x_1}|\,r}|\;\overline{\overline{x_2}|\,r}|\ldots\overline{\overline{x_m}|\,r}|}|}|$$
$$= \overline{\overline{\overline{\overline{\overline{\overline{a_1}|}|\;\overline{\overline{a_2}|}|\ldots\overline{\overline{a_n}|}|}|\;\overline{r}|}|\;\overline{\overline{\overline{\overline{x_1}|}|\;\overline{\overline{x_2}|}|\ldots\overline{\overline{x_m}|}|}|\;r}|}|}| \qquad \text{(J2.1* twice)}$$
$$= \overline{\overline{\overline{\overline{a_1a_2\ldots a_n}|\;\overline{r}|}|\;\overline{\overline{x_1x_2\ldots x_m}|\;r}|}|}| \qquad \text{(C1 } n{+}m \text{ times)}$$
$$= \overline{\overline{r}|\,a_1a_2\ldots a_n}|\;\overline{rx_1x_2\ldots x_m}| \qquad \text{(C9.1)}$$

□

Next we prove a generalizion of C2.

Theorem (C2*).

$$\overline{\overline{\overline{\overline{a_nb}|\ldots}|\;a_2}|\;a_1}|\,b = \overline{\overline{\overline{\overline{a_n}|\ldots}|\;a_2}|\;a_1}|\,b$$

Proof. The proof proceeds by induction on n. The base case is C2, where $n = 1$. Let the induction hypothesis be:

$$\overline{\overline{\overline{\overline{a_nb}|\ldots}|\;a_2}|\;a_1}|\,b = \overline{\overline{\overline{\overline{a_n}|\ldots}|\;a_2}|\;a_1}|\,b$$

Substitute $\overline{a_{n+1}b}|\;a_n$ for a_n . The induction step then follows immediately:

$$\overline{\overline{\overline{\overline{\overline{a_{n+1}b}|\;a_nb}|\ldots}|\;a_2}|\;a_1}|\,b = \overline{\overline{\overline{\overline{\overline{a_{n+1}b}|\;a_n}|\ldots}|\;a_2}|\;a_1}|\,b$$

□

Spencer-Brown does not mention a generalized C7. Here is one possible version.

Theorem (C7*). *Let n be a positive even number. Then for all such n the following pair of equations holds:*

(i) $\overline{\overline{\overline{\overline{a_n}|\ldots}|\;a_2}|\;a_1}| \;=\; \overline{\overline{a_n}|\,a_{n-1}\ldots a_3a_1}|\ldots\overline{\overline{a_4}|\;a_3a_1}|\;\overline{\overline{a_2}|\;a_1}|$

(ii) $\overline{\overline{\overline{\overline{\overline{a_{n+1}}|\;a_n}|\ldots}|\;a_2}|\;a_1}| \;=\; \overline{a_{n+1}a_{n-1}\ldots a_3a_1}|\;\overline{\overline{a_n}|\,a_{n-1}\ldots a_3a_1}|\ldots\overline{\overline{a_4}|\;a_3a_1}|\;\overline{\overline{a_2}|\;a_1}|$

Proof. Let equation (i) be the induction hypothesis. The base case is the identity $\overline{\overline{a_2}\rceil\, a_1}\rceil = \overline{\overline{a_2}\rceil\, a_1}\rceil$, where $n = 2$. Now substitute $\overline{a_{n+1}}\rceil\, a_n$ for a_n. Then,

$$
\begin{aligned}
&\overline{\overline{\overline{\overline{\overline{a_{n+1}}\rceil\, a_n}\rceil \ldots}\rceil\, a_2}\rceil\, a_1}\rceil \\
&= \overline{\overline{\overline{a_{n+1}}\rceil\, a_n}\rceil\, a_{n-1}\ldots a_3a_1}\rceil \ldots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\, \overline{\overline{a_2}\rceil\, a_1}\rceil && \text{(i)} \\
&= \overline{a_{n+1}a_{n-1}\ldots a_3a_1}\rceil\, \overline{\overline{a_n}\rceil\, a_{n-1}\ldots a_3a_1}\rceil \ldots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\, \overline{\overline{a_2}\rceil\, a_1}\rceil && \text{(C7)}
\end{aligned}
$$

proving the implication from (i) to *(ii)*. In equation (ii) substitute $\overline{a_{n+2}}\rceil\, a_{n+1}$ for a_{n+1}. Then,

$$
\overline{\overline{\overline{\overline{\overline{a_{n+2}}\rceil\, a_{n+1}}\rceil \ldots}\rceil\, a_2}\rceil\, a_1}\rceil = \overline{\overline{a_{n+2}}\rceil a_{n+1}\ldots a_3a_1}\rceil \ldots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\, \overline{\overline{a_2}\rceil\, a_1}\rceil \qquad \text{(ii)}
$$

proving (i) for the succeeding even number. This proves the proposition for all $n \geq 2$, and hence for all echelons of depth greater than or equal to 2. □

Theorem (T14). *Any expression can be reduced to an equivalent expression not more than two crosses deep. Specifically, any expression E is equivalent to $\overline{\overline{a_1}\rceil\, b_1}\rceil\, \overline{\overline{a_2}\rceil\, b_2}\rceil \ldots \overline{\overline{a_n}\rceil\, b_n}\rceil\ \overline{c_1}\rceil\, \overline{c_2}\rceil \ldots \overline{c_m}\rceil\, d$ where a_i, b_i, c_i, d are composed (at most) of juxtapositions of variables and the two constants, $\overline{}\rceil$ and .*

Proof. Repeated applications of **C7*** to any expression demonstrates the theorem. Spencer-Brown uses **C7** (not having proven a generalization), but it comes to the same thing. □

The final theorem follows Spencer-Brown closely.

Theorem (T15). *Given any expression E and any variable v, E can be reduced to an equivalent expression containing not more than two appearances of v.*

Proof. In the case where v is not in E, the theorem is trivially true, since $E = \overline{\overline{v}\rceil\, v}\rceil\, E$ by **J1**. So let us suppose that v appears in E. Using **C7*** as many times as necessary, we rewrite E:

$$
E = \overline{\overline{va_1}\rceil\, b_1}\rceil\, \overline{\overline{va_2}\rceil\, b_2}\rceil \ldots \overline{\overline{va_n}\rceil\, b_n}\rceil\ \overline{vc_1}\rceil\, \overline{vc_2}\rceil \ldots \overline{vc_m}\rceil\, d
$$

where a_i, b_i, c_i, and d are expressions free of v. Then, by n applications of **C8.1**,

$$
\begin{aligned}
E &= \overline{\overline{v}\rceil\, b_1}\rceil\, \overline{\overline{a_1}\rceil\, b_1}\rceil\, \overline{\overline{v}\rceil\, b_2}\rceil\, \overline{\overline{a_2}\rceil\, b_2}\rceil \ldots \overline{\overline{v}\rceil\, b_n}\rceil\ \overline{\overline{a_n}\rceil\, b_n}\rceil\, \overline{vc_1}\rceil\, \overline{vc_2}\rceil \ldots \overline{vc_m}\rceil\, d \\
&= \overline{\overline{v}\rceil\, b_1}\rceil\, \overline{\overline{v}\rceil\, b_2}\rceil \ldots \overline{\overline{v}\rceil\, b_n}\rceil\ \overline{vc_1}\rceil\, \overline{vc_2}\rceil \ldots \overline{vc_m}\rceil\, f \\
&\qquad \text{(where } f = \overline{\overline{a_1}\rceil\, b_1}\rceil\, \overline{\overline{a_2}\rceil\, b_2}\rceil \ldots \overline{\overline{a_n}\rceil\, b_n}\rceil\, d \text{ is free of } v.) \\
&= \overline{\overline{\overline{b_1}\rceil\, \overline{b_2}\rceil \ldots \overline{b_n}\rceil}\rceil\, \overline{v}\rceil}\rceil\, \overline{\overline{\overline{c_1}\rceil\, \overline{c_2}\rceil \ldots \overline{c_m}\rceil}\rceil\, v}\rceil\, f && \text{(J2.1* twice)}
\end{aligned}
$$

□